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On the viscous hypersonic blunt body problem

By WILLIAM B. BUSH
California Institute of Technology, Pasadena, California

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The viscous hypersonic flow past an axisymmetric blunt body is analysed based upon the Navier–Stokes equations. It is assumed that the fluid is a perfect gas having constant specific heats, a constant Prandtl number, P, whose numerical value is of order one, and a viscosity coefficient varying as a power, ω, of the absolute temperature. Limiting forms of solutions are studied as the free-stream Mach number, M, and the free-stream Reynolds number based on the body nose radius, R, go to infinity, and $\epsilon = (\gamma-1)/(\gamma+1)$, where γ is the ratio of the specific heats, and $\delta = 1/(\gamma-1)M^2$ go to zero.

Through the use of asymptotic expansions and matching, it is shown that three distinct regions comprise the interior of the 'shock structure', and that one, two or three regions make up the 'shock layer', depending on whether the quantity $R\delta^\omega$ is of order ϵ^{-1}, $\epsilon^{-\frac{3}{2}}$ or $\epsilon^{-n}(n > \frac{3}{2})$, respectively, as the various limits are approached. The behaviour of the flow in these regions is partly analysed.

1. General description of problem and results

The aim of this paper is to present a brief account of the problem treated fully by Bush (1964) with emphasis on the methods used and the results achieved.

In high Mach number, high altitude flight, the Reynolds number, although still large, may be sufficiently low for viscous interactions and the structure of the viscous layers to become important. This problem is analysed in the present paper on the basis of the Navier–Stokes equations with the idealizations that the fluid is a perfect gas having constant specific heats, constant Prandtl number, $P = O(1)$, and viscosity coefficient varying as a power, ω, of the absolute temperature. In order to simplify the equations in a way appropriate to this problem, asymptotic expansions of the Navier–Stokes equations are constructed as the free-stream Mach number, $M = U_\infty/(\gamma p_\infty/\rho_\infty)^{\frac{1}{2}}$, and the free-stream Reynolds number, $R = \rho_\infty U_\infty a/\mu_\infty$, go to infinity. Further, the assumption of Newtonian flow theory that the density ratio is large across the bow shock wave is used and is essential for achieving simplification of the equations. The Newtonian parameter, $\epsilon = (\gamma-1)/(\gamma+1)$, goes to zero in such a way that

$$\delta = 1/(\gamma-1)M^2 = (1-\epsilon)/2\epsilon M^2$$

also goes to zero.

This is essentially the same starting-point as in the analyses of Hayes & Probstein (1959), Cheng (1961, 1963), and others. This analysis shows that there are actually three distinct regions within the 'shock structure', rather than the two regions proposed by Cheng (1963). Further, it is shown that there are three distinct régimes to the make-up of the 'shock layer', not four, as suggested by Hayes & Probstein. To show this requires a great deal of manipulation of complicated expansions. The essential features of this mathematics are presented in §2, but a great deal, including the 'matchings' of asymptotic expansions, has been suppressed in order to keep this paper concise. The reader who wishes to pursue the details further should refer to Bush (1964), from which this paper is extracted, and Bush (1962), which extensively treats the concept of 'matching' for a similar yet simpler problem.

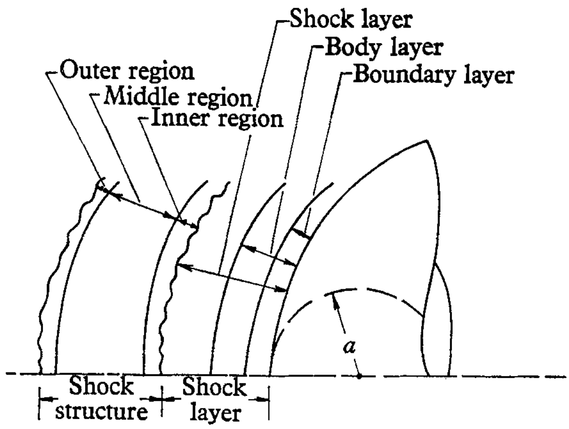

FIGURE 1. Regions of the flow.

The basic idea is that orders of magnitude for various terms in the equations are found so that the simplified equations are consistently valid in regions of reduced size; a criterion of consistency is the ability of a solution to join on to solutions in neighbouring regions, i.e. to satisfy matching and boundary conditions.

The same method could be applied to various more realistic versions of the Navier–Stokes equations, with some increase in complexity. However, this paper takes no steps in the direction of rarefied gas flow.

An outline is now given of the results obtained by systematic use of asymptotic expansions. A natural starting-point is a discussion of approximations to shock wave structure. A schematic picture of the flow is given in figure 1.

The methods used here are closely related to singular-perturbation techniques used by Bush (1962) to study one-dimensional shock wave structure when M goes to infinity for the same conditions as prescribed above with the important exceptions that ϵ is fixed, P is $\frac{3}{4}$, and the viscosity obeys the Sutherland law. Further, these ideas can be used to obtain the structure of the detached shock wave that is supported by an axisymmetric blunt-body in a steady supersonic uniform stream when M and R go to infinity, in such a way that $M^{2\omega}/R$ goes to zero, the viscosity obeying the power law rather than the Sutherland law. In this limit, the shock layer remains finite with its thickness a sizeable fraction of the body nose radius a. The flow in the shock layer must be found by solving the full inviscid equations of motion (e.g. Van Dyke 1958). It is found, as expected, that the shock structure is locally the same as that of the normal shock. In the present paper, since ϵ goes to zero, the shock layer also becomes thin (the shock-layer thickness divided by the nose radius of the body is $O(\epsilon)$) so that the shock structure is modified. Furthermore, this additional limit allows the flow in the shock layer, at least in part, to be determined analytically.

In the solution of the shock structure as $M \to \infty$ with ϵ fixed, there are necessarily two regions to the shock structure, where the behaviour of the flow quantities is described by two distinct sets of asymptotic expansions. One region is the very thin outer region, whose ratio of thickness to body nose radius is $O(1/R) \to 0$. The orders of magnitude of the flow quantities in this region are those for the quantities in the free stream. The second region is the relatively thicker inner, or principal, region, which has a thickness in units of nose radius (thickness ratio) of $O(M^{2\omega}/R) \to 0$. The velocity components and the density here are of the same order of magnitude as in the free stream, but the temperature and pressure in this region divided by their free-stream values are of $O(M^2) \to \infty$. These two sets of asymptotic expansions for the shock structure are shown to be the correct ones by proving that the expansions for outer and inner regions and the expansions for the inner region and the shock layer match in intermediate regions of common validity as well as satisfy the boundary conditions.

It is found in this paper that, with $\epsilon \to 0$, there are now three regions to the shock structure and, hence, three distinct sets of asymptotic expansions are required to describe the behaviour of the flow quantities in the shock structure.

The first of these three regions is the outer region. With Bush (1962) as a guide, it is natural to postulate that a region should exist in the outer portion of the shock structure, adjacent to the uniform upstream region to guarantee the uniformity of the solution at upstream 'infinity'. This region may be thought of as acting as a very thin transition zone between the relatively cool free stream and the hot major (middle) region of the shock structure. In the outer region the order of magnitude of the flow quantities is characterized by their magnitude in the free stream. The leading terms in the expansions for this dissipationless outer region and the solution of the equations of motion for these leading terms for this region are presented in §2. Among other things, it is seen that the thickness ratio for the outer region is $O(1/R) \to 0$, just as in the ϵ-fixed analysis.

There must be two distinct shock structure regions between the outer region and the shock layer in the $\epsilon \to 0$ problem (rather than just one, as in the ϵ-fixed problem) because there is no single asymptotic expansion which will match to both the expansions of the outer region and to those of the shock layer as $\epsilon \to 0$. There are, however, two distinguished regions, called in this paper the middle and inner regions, whose sets of expansions permit complete matching (i.e. outer region-middle region, middle region-inner region, and inner region-shock layer matching) in the limit as $\epsilon \to 0$.

The middle, or dissipation, region is also a thin region (although a thicker one than either the outer or inner region) with a thickness ratio of $O(1/R\delta^{\omega}) \to 0$. This dissipation zone combines with the outer region to be essentially the 'shock-transition zone' treated by Cheng (1963). The velocity ratios, (u/U_∞) and (v/U_∞), and the density ratio, (ρ/ρ_∞), are all of $O(1)$, but the temperature and pressure ratios, (T/T_∞) and (p/p_∞), are $O(1/\delta) \to \infty$. Again, the solutions for the leading terms in the equations of motion are given in §2.

The solutions presented in §2 for the outer and middle regions, and all the solutions for the succeeding regions to be presented, are, of course, found only by matching between adjacent regions. Thus, the procedure is to propose tentative solutions for each of the two adjacent regions based on the physics of the problem, and then verify that the proposed solutions are valid by showing that they match in an 'overlapping zone' that is between the two regions under consideration.

For the outer region-middle region matching, the outer region expansions have a certain behaviour as the outer region variable, v_0, goes to infinity, which must match to the middle region expansions as the middle region variable, v_m, goes to $(-\sin\Phi)$; Φ is defined in figure 2. It is found that all the required matchings between the outer and middle regions can be performed under the restriction that the Prandtl number be greater than $\frac{1}{2}$ but less than $\frac{3}{2}$.

The inner region is the thin, dissipationless transition zone between the middle region of the shock structure and the shock layer. In this region, which is the 'neighbourhood of the shock interface' alluded to by Cheng (1963), in which his 'shock-transition zone' equations are not strictly valid, there is a decrease in the magnitude of the normal velocity, v, and a corresponding increase in the density; the temperature is the same order of magnitude as in the middle region; but the pressure increases due to the increase in the density. In terms of the dimensionless ratios introduced, the thickness ratio of the inner region is $O(\epsilon/R\delta^{\omega})\rightarrow 0$, and (u/U_∞) is $O(1)$, (v/U_∞) is $O(\epsilon)\rightarrow 0$, while (ρ/ρ_∞) is $O(1/\epsilon)\rightarrow\infty$, (T/T_∞) is $O(1/\delta)\rightarrow\infty$, and (p/p_∞) is $O(1/\epsilon\delta)\rightarrow\infty$. The solutions for the leading terms in the equations of motion in this region, consistent with the matching with the middle region are given in §2.

The matching of the middle and inner regions is, again, done with respect to the normal velocity, v, and middle region expansions as v_m goes to zero match to the inner region expansions as $|v_i|$ goes to infinity.

It should be pointed out that, in solving for the flows in the different regions of the shock structure, there are quantities in each region which are not completely determined until the flow in the region just interior to the one under consideration is known. This means that the flow in the shock structure is not completely known until the flow in the shock layer itself is known.

Next consider the flow between the shock wave and the body, in the 'shock layer'. Different structures for this layer arise depending on the 'similarity' parameter $K = (\epsilon R\delta^{\omega})^{-1}$.* This parameter measures the rate at which $R\rightarrow\infty$ compared with $(\epsilon,\delta)\rightarrow 0$. The quantity K = const. represents a similar family of flows. The parameter $K\rightarrow 0$ at various rates represents other families of flows.

The shock layer has a thickness ratio that is $O(\epsilon)\rightarrow 0$ and the magnitudes of the flow quantities in the layer are the same as those in the inner region of the shock structure (see §2). In the equations of motion for this layer the ratio of the viscous and heat conduction contributions to the inviscid contributions is $O(K)$. Thus, if $K\rightarrow 0$ the shock layer is an inviscid one and the (inviscid) Rankine–Hugoniot shock relations as $M\rightarrow\infty$ and $\epsilon\rightarrow 0$, with $\epsilon M^2\rightarrow\infty$, are the proper

* $2^{\omega}/(1+2\delta)^{\omega}K$ may be identified with the K_0^2 used by Cheng.

boundary conditions at the outer edge of the shock layer. On the other hand, if K is $O(1)$, then the entire shock layer is a viscous shock layer and the boundary conditions at the outer edge of the viscous shock layer are not the Rankine–Hugoniot shock relations, but rather are ones in which the viscous and heat conduction terms right behind the shock wave are important. These shock relations are given in §2. Note that the alternative of $K \to \infty$ is ruled out as not being physically realistic.

Of course, the boundary conditions for the outer edge of the shock layer, in either case, $K \leqslant O(1)$, are determined by matching between the inner region of the shock structure and the shock layer. The matching in this case is most easily accomplished by matching with respect to the normal co-ordinate y, and the inner region expansions as $\eta_i \to \infty$ match with the shock layer expansions as $\eta_L \to \mathscr{Y}_L$, the outer edge of the shock layer.

For the viscous shock conditions, the results obtained can be shown to be essentially those given by Cheng (1961). Further, it should be noted here that the use of asymptotic expansions for the shock structure provides the theoretical basis for the absence of the shock structure's thickness-curvature effects.

In the terminology of Hayes & Probstein (1959), the viscous flow régime just described is the 'incipient merged layer' régime. It should be emphasized that the inviscid shock-layer equations must be solved using the inviscid outer edge conditions, and the viscous shock-layer equations must be solved using the viscous outer edge conditions. This rules out the 'viscous layer' régime, in which the viscous shock-layer equations are solved subject to the Rankine–Hugoniot relations at the outer edge.

It should be noted that, since the ratio of the thickness of the shock structure to the thickness of the shock layer is $O(K)$, the shock structure is thinner than the inviscid shock layer, but grows until its thickness is of the same order of magnitude as the shock layer thickness when the shock layer is viscous.

The complete solution for the flow in a viscous shock layer (i.e. $K = O(1)$ and the viscous shock layer makes up the entire shock layer), due to the complexity of the partial differential equations of motion for such a layer, was felt to be beyond the scope of this investigation. However, due to the geometrical symmetry of the problem and the fact that the partial differential equations are parabolic in type, the flow in the vicinity of the axis of symmetry can be found by solving a set of ordinary differential equations.

For the special case $\omega = 1$, these ordinary differential equations uncouple to such an extent that their solutions can be found in terms of tabulated functions, as shown first by Cheng (1961). For a general value of ω, however, one must resort to numerical computation of the solution, although the nature of the equations is such that the two-point boundary-value problem can be transformed into what is effectively a one-point boundary-value problem.

When $K \to 0$, the complete shock layer consists of the inviscid shock layer plus the subregions necessary to satisfy the viscous boundary conditions at the body surface. The complete solution for the flow in the inviscid shock layer itself is already known, having been found by Freeman (1956) in terms of modified von Mises variables. The complete solution is also presented in §2, but in terms of modified Crocco variables, because these variables are found to be more suitable for treating the inviscid shock layer and the connected subregions of the shock layer.

In the case of the inviscid shock layer ($u/U_\infty = O(1)$), it is found that the solutions, which are based on orders of magnitude which hold at the shock wave, are not valid right up to the body surface (cf. Chester 1956). To remove this difficulty near the body surface it is necessary to introduce a body sublayer, imbedded in the inviscid shock layer close to the body surface, in which the order of magnitude of u is some small fraction of U_∞. The proper body layer is the one for which the thickness ratio is $O(\epsilon^{\frac{3}{2}})$, the velocity components (u/U_∞) and (v/U_∞) are $O(\epsilon^{\frac{1}{2}})$ and $O(\epsilon^2)$, respectively, (ρ/ρ_∞) is $O(1/\epsilon)$, (T/T_∞) is $O(1/\delta)$, and (p/p_∞) is $O(1/\epsilon\delta)$. The existence of such a layer is, of course, verified by showing that this layer matches with the inviscid shock layer and has the proper behaviour at the body.

The structure of this body layer depends on a second similarity parameter. In the body layer the ratio of the viscosity and heat conduction contributions to the inviscid contributions is $D = (\epsilon^{\frac{5}{2}} R\delta^\omega)^{-1}$. Therefore, the body layer is inviscid if $D \to 0$ and viscous if $D = O(1)$. In the terminology of Hayes & Probstein (1959) the viscous body layer is the 'vorticity interaction' layer.

A Crocco transformation similar to that used in the inviscid shock layer is used in the body layer in order to match with the shock layer. The matching takes place with respect to t, where $t = u/(U_\infty \sin \Phi)$, i.e. the body-layer expansions as t_c goes to infinity match with the inviscid shock layer expansions as t_L goes to zero. The boundary conditions at the outer edge of the body layer, which are the same whether the layer is inviscid or viscous, are given in §2.

Thus, when $D = O(1)$ (and $K \to 0$), the complete shock layer consists of the inviscid shock layer and the viscous body layer. As before, ordinary differential equations can be derived which are valid for the flow in the vicinity of the stagnation point. However, work concerning the solutions of such equations, which would yield the shear and heat transfer at the nose of the body, is not presented here.

On the other hand, for $D \to 0$ (and $K \to 0$), the complete shock layer is made up of the inviscid shock layer, the inviscid body layer, and the 'classical' viscous boundary layer. The latter, of course, is necessary in order to satisfy the viscous boundary conditions at the body. The solution for the leading terms for the flow quantities in the inviscid body layer is presented in §2 in terms of the proper modified Crocco variables.

The 'classical' viscous boundary layer that is imbedded within the inviscid body layer is the last layer to be discussed. For such a boundary layer the thickness ratio is $O((\epsilon^{\frac{1}{2}}/R\delta^\omega)^{\frac{1}{2}}) \to 0$, and the flow quantities are

$$(u/U_\infty) = O(\epsilon^{\frac{1}{2}}), \quad (v/U_\infty) = O((\epsilon^{\frac{3}{2}}/R\delta^\omega)^{\frac{1}{2}}) \to 0,$$

and

$$(\rho/\rho_\infty) = O(1/\epsilon), \quad (T/T_\infty) = O(1/\delta), \quad (p/p_\infty) = O(1/\epsilon\delta),$$

as shown in §2. Note that the ratio of the boundary-layer thickness to the inviscid body-layer thickness is $D^{\frac{1}{2}}$, and, by definition, $D \to 0$ for the inviscid body layer.

The matching between this viscous boundary layer and the inviscid body layer takes place with respect to the normal co-ordinate, y. The boundary-layer expansions as η_{BL} goes to infinity match with the inviscid body layer as η_c goes to zero. The boundary conditions at the outer edge of the boundary layer, determined by this matching are presented in §2.

Table 1 summarizes the results just described and the orders of magnitude for the skin-friction $\bar{C}_f = \mu_w(\partial u/\partial y)_w/(B\mu_\infty U_\infty/a^2)$, and the heat transfer coefficient, $\bar{C}_h = k_w(\partial T/\partial y)_w/(k_\infty T_\infty/a)$.

$$K = \frac{1}{\epsilon R\delta^\omega},\quad D = \frac{1}{\epsilon^{\frac{5}{2}}R\delta^\omega},\quad \bar{C}_f = \frac{(\mu\partial u/\partial y)_w}{B\mu_\infty U_\infty/a^2},\quad \bar{C}_h = \frac{(k\partial T/\partial y)_w}{k_\infty T_\infty/a}$$

$K = O(1)$ $D \to \infty$	Viscous shock layer	$\bar{C}_f = O(1/\epsilon\delta^\omega)$ $\bar{C}_h = O(1/\epsilon\delta^{1+\omega})$
$K \to 0$ $D = O(1)$	Inviscid shock layer + viscous body layer	$\bar{C}_f = O(1/\epsilon\delta^\omega)$ $\bar{C}_h = O(1/\epsilon^{\frac{3}{2}}\delta^{1+\omega})$
$K \to 0$ $D \to 0$	Inviscid shock layer + inviscid body layer + viscous boundary layer	$\bar{C}_f = O(1/D^{\frac{1}{2}}\epsilon\delta^\omega)$ $\bar{C}_h = O(1/D^{\frac{1}{2}}\epsilon^{\frac{3}{2}}\delta^{1+\omega})$

TABLE 1.

In the next section the equations of motion and the forms of the asymptotic expansions are presented.

2. Expansions for the regions

2.1. *The equations of motion*

Non-dimensional variables are used, all lengths being referred to a, the body nose radius; all velocities to U_∞, the free-stream speed; and the pressure, density, temperature, and viscosity to their free-stream values. Thus, the notation being as in figure 2, we make the following replacements:

$$(x/a, y/a) \to (x, y), \quad (B/a, a\kappa) \to (B, \kappa), \quad (u/U_\infty, v/U_\infty) \to (u, v)$$

and $$(p/p_\infty, \rho/\rho_\infty, T/T_\infty, \mu/\mu_\infty) \to (p, \rho, T, \mu).$$

Further, let $\xi = x$, $\eta = y - Y$, where Y is the non-dimensional measure of the distance from the body to the region under discussion. Then, the equations of

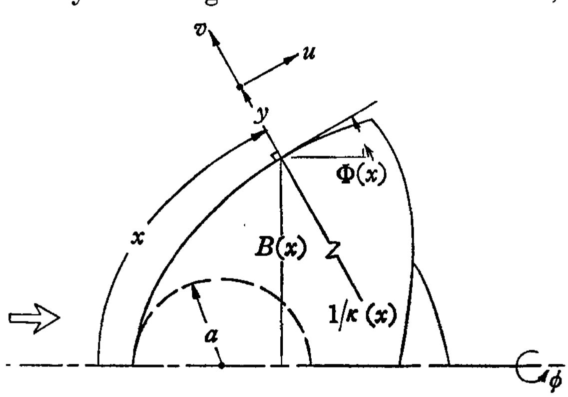

FIGURE 2. Notation.

continuity, tangential and normal momentum, energy, state, and the viscosity law are equations (1)–(6), respectively, expressed in terms of these non-dimensional quantities. These equations comprise the Navier–Stokes system to be solved with the uniform conditions at upstream 'infinity' and the no-slip and temperature conditions at the body surface.

$$\frac{\partial(\rho v)}{\partial\eta}+\frac{1}{h}\frac{\partial(\rho u)}{\partial x}+\frac{\kappa\rho v}{h}+\frac{\rho(u\sin\Phi+v\cos\Phi)}{r}=0, \tag{1}$$

$$\begin{aligned}\rho\left(\frac{u}{h}\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial\eta}+\frac{\kappa uv}{h}\right)+\frac{1-\epsilon}{1+\epsilon}\frac{1}{M^2}\frac{1}{h}\frac{\partial p}{\partial x}\\=\frac{1}{R}\Bigg[\left(\frac{\partial}{\partial\eta}+\frac{2\kappa}{h}+\frac{\cos\Phi}{r}\right)\left(\mu\left[\frac{\partial u}{\partial\eta}-\frac{\kappa u}{h}+\frac{1}{h}\frac{\partial v}{\partial x}\right]\right)\\+\frac{2}{3}\frac{1}{h}\frac{\partial}{\partial x}\left(\mu\left[\frac{2}{h}\left\{\frac{\partial u}{\partial x}+\kappa v\right\}-\frac{\partial v}{\partial\eta}-\frac{u\sin\Phi+v\cos\Phi}{r}\right]\right)\\+\frac{2\sin\Phi}{r}\left(\mu\left[\frac{1}{h}\left\{\frac{\partial u}{\partial x}+\kappa v\right\}-\frac{u\sin\Phi+v\cos\Phi}{r}\right]\right)\Bigg],\end{aligned} \tag{2}$$

$$\begin{aligned}\rho\left(\frac{u}{h}\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial\eta}-\frac{\kappa u^2}{h}\right)+\frac{1-\epsilon}{1+\epsilon}\frac{1}{M^2}\frac{\partial p}{\partial\eta}\\=\frac{1}{R}\Bigg[\left(\frac{4}{3}\frac{\partial}{\partial\eta}+\frac{2\kappa}{h}+\frac{2\cos\Phi}{r}\right)\left(\mu\frac{\partial v}{\partial\eta}\right)\\+\left(\frac{1}{h}\frac{\partial}{\partial x}+\frac{\sin\Phi}{r}\right)\left(\mu\left[\frac{\partial u}{\partial\eta}-\frac{\kappa u}{h}+\frac{1}{h}\frac{\partial v}{\partial x}\right]\right)\\-\left(\frac{2}{3}\frac{\partial}{\partial\eta}+\frac{2\kappa}{h}\right)\left(\mu\left[\frac{1}{h}\left\{\frac{\partial u}{\partial x}+\kappa v\right\}\right]\right)\\-\left(\frac{2}{3}\frac{\partial}{\partial\eta}+\frac{2\cos\Phi}{r}\right)\left(\mu\left[\frac{u\sin\Phi+v\cos\Phi}{r}\right]\right)\Bigg],\end{aligned} \tag{3}$$

$$\begin{aligned}\rho\left(\frac{u}{h}\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial\eta}\right)-\frac{2\epsilon}{1+\epsilon}\left(\frac{u}{h}\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial\eta}\right)\\=\frac{1}{PR}\Bigg[\frac{\partial}{\partial\eta}\left(\mu\frac{\partial T}{\partial\eta}\right)+\left(\frac{\kappa}{h}+\frac{\cos\Phi}{r}\right)\mu\frac{\partial T}{\partial\eta}\\+\frac{1}{h}\frac{\partial}{\partial x}\left(\frac{\mu}{h}\frac{\partial T}{\partial x}\right)+\frac{\sin\Phi}{r}\frac{\mu}{h}\frac{\partial T}{\partial x}\Bigg]\\+\frac{2\epsilon M^2}{1-\epsilon}\frac{\mu}{R}\Bigg[2\left(\frac{\partial v}{\partial\eta}\right)^2+2\left(\frac{1}{h}\left\{\frac{\partial u}{\partial x}+\kappa v\right\}\right)^2\\+2\left(\frac{u\sin\Phi+v\cos\Phi}{r}\right)^2+\left(\frac{\partial u}{\partial\eta}-\frac{\kappa u}{h}+\frac{1}{h}\frac{\partial v}{\partial x}\right)^2\\-\frac{2}{3}\left(\frac{1}{h}\frac{\partial u}{\partial x}+\frac{\partial v}{\partial\eta}+\frac{\kappa v}{h}+\frac{u\sin\Phi+v\cos\Phi}{r}\right)^2\Bigg],\end{aligned} \tag{4}$$

$$p=\rho T, \tag{5}$$

$$\mu=T^{\omega}, \tag{6}$$

where $$h = 1+\kappa(Y+\eta) \quad \text{and} \quad r = B+(Y+\eta)\cos\Phi,$$

and $$(\partial/\partial x) = (\partial/\partial\xi) - Y'(\xi)\,(\partial/\partial\eta).$$

2.2 *The uniform upstream region*

In the uniform upstream region the flow quantities are

$$p = \rho = T = 1, \quad u = \cos\Phi, \quad v = -\sin\Phi. \tag{7}$$

2.3. *The outer region of the shock structure*

The co-ordinates for the outer region are

$$x = \xi = \xi_0, \quad y = Y(\xi)+\eta = \epsilon Y_0(\xi_0) + (1/R)\,\eta_0, \tag{8}$$

and the leading terms in the expansions for the flow quantities in terms of the small parameter δ are

$$\left.\begin{aligned} u &= \cos\Phi + \delta^{1/P}u_0 + \dots, \quad v = -\sin\Phi + \delta^{3/4P}v_0 + \dots, \\ \rho &= 1+\delta^{3/4P}\rho_0 + \dots, \quad p = p_0 + \dots, \quad T = T_0 + \dots. \end{aligned}\right\} \tag{9}$$

The first-order equations of motion, based upon these expansions, with ξ_0 and v_0 as the independent variables, are

$$\left.\begin{aligned} & p_0 = T_0, \quad \rho_0 = v_0/\sin\Phi(\xi_0), \\ & \frac{\partial\eta_0}{\partial v_0} + \frac{4}{3}\frac{T_0^\omega}{v_0\sin\Phi(\xi_0)} = 0, \\ & \frac{\partial u_0}{\partial v_0} - \frac{4}{3}\frac{u_0}{v_0} = \{\epsilon/\delta^{1/4P}\}\tfrac{1}{3}Y_0'(\xi_0), \\ & \frac{\partial(T_0-1)}{\partial v_0} - \frac{4P}{3}\frac{(T_0-1)}{v_0} = 0. \end{aligned}\right\} \tag{10}$$

Thus, the flow quantities in the outer region may be written as

$$\left.\begin{aligned} u_0(\xi_0, v_0) &= w_0(\xi_0)\,v_0^{\frac{4}{3}} - \{\epsilon/\delta^{1/4P}\}\,Y_0'(\xi_0)\,v_0, \\ p_0(\xi_0, v_0) &= T_0(\xi_0, v_0) = 1 + S_0(\xi_0)\,v_0^{4P/3}, \\ \rho_0(\xi_0, v_0) &= v_0/\sin\Phi(\xi_0), \\ \eta_0(\xi_0, v_0) &= \frac{4}{3\sin\Phi(\xi_0)}\int_{A_0}^{v_0}\frac{[T_0(\xi_0,\nu_0)]^\omega\,d\nu_0}{\nu_0}, \end{aligned}\right\} \tag{11}$$

where $w_0(\xi_0)$ and $S_0(\xi_0)$ are functions that are determined only after matching with the solution of the complete shock structure.

2.4. *The middle region of the shock structure*

For the middle region

$$x = \xi = \xi_m, \quad y = Y(\xi)+\eta = \epsilon Y_m(\xi_m) + (1/R\delta^\omega)\,\eta_m, \tag{12}$$

$$\left.\begin{aligned} & u = u_m + \dots, \quad v = v_m + \dots, \\ & \rho = \rho_m + \dots, \quad p = (1/\delta)\,p_m + \dots, \quad T = (1/\delta)\,T_m + \dots. \end{aligned}\right\} \tag{13}$$

The equations associated with these expansions are

$$\left.\begin{aligned}
&p_m = \rho_m T_m, \quad \rho_m v_m = -\sin\Phi\,(\xi_m),\\
&\frac{\partial U_m}{\partial V_m} - \frac{4}{3}\frac{U_m}{V_m} = 0, \quad U_m = u_m - \cos\Phi\,(\xi_m), \quad V_m = v_m + \sin\Phi\,(\xi_m),\\
&\frac{\partial^2 T_m}{\partial V_m^2} - \frac{(\frac{4}{3}P-1)}{V_m}\frac{\partial T_m}{\partial V_m} + P\left[\frac{4}{3} + \left(\frac{\partial U_m}{\partial V_m}\right)^2\right] = 0.\\
&\frac{\partial \eta_m}{\partial V_m} + \frac{4}{3\sin\Phi}\frac{T_m^\omega}{V_m} = 0.
\end{aligned}\right\} \tag{14}$$

The solutions of these equations are

$$\left.\begin{aligned}
u_m(\xi_m, v_m) &= \cos\Phi(\xi_m) + w_m(\xi_m)\,[\sin\Phi\,(\xi_m) + v_m]^{\frac{4}{3}},\\
\rho_m(\xi_m, v_m) &= -\sin\Phi\,(\xi_m)/v_m,\\
T_m(\xi_m, v_m) &= S_m(\xi_m)\,[\sin\Phi + v_m]^{4P/3} - \frac{P}{3-2P}[\sin\Phi + v_m]^2\\
&\qquad - \frac{Pw_m^2}{2(2-P)}[\sin\Phi + v_m]^{\frac{8}{3}},\\
p_m(\xi_m, v_m) &= [\rho_m(\xi_m, v_m)]\,[T_m(\xi_m, v_m)],\\
\eta_m(\xi_m, v_m) &= \frac{4}{3\sin\Phi}\int_{v_m}^{0}\frac{[T_m(\xi_m, \nu_m)]^\omega\, d\nu_m}{[\sin\Phi + \nu_m]}.
\end{aligned}\right\} \tag{15}$$

Again, the quantities $w_m(\xi_m)$ and $S_m(\xi_m)$ depend on the solution of the complete shock structure.

2.5. *The inner region of the shock structure*

The quantities in the inner region have the following representation:

$$x = \xi = \xi_i, \quad y = Y(\xi) + \eta = \epsilon Y_i(\xi_i) + (\epsilon/R\delta^\omega)\,\eta_i, \tag{16}$$

$$\left.\begin{aligned}
u = W(\xi_i) + \epsilon u_i + \ldots, \quad v = \epsilon v_i + \ldots,\\
\rho = (1/\epsilon)\rho_i + \ldots, \quad p = (1/\epsilon\delta)\,p_i + \ldots, \quad T = (1/\delta)\,[\Theta(\xi_i) + \epsilon T_i + \ldots].
\end{aligned}\right\} \tag{17}$$

Introducing the quantity $V_i = v_i - Y_i'(\xi_i)\,W(\xi_i)$, the simplified equations of motion for the region and their solutions are

$$\left.\begin{aligned}
&p_i = \rho_i\Theta, \quad \rho_i V_i = -\sin\Phi,\\
&-2\sin\Phi\,(\Theta/V_i) - \tfrac{4}{3}\Theta^\omega\,(\partial V_i/\partial\eta_i) = \sin^2\Phi,\\
&\Theta^\omega(\partial u_i/\partial\eta_i) = \sin\Phi\,(\cos\Phi - W),\\
&\frac{\Theta^\omega}{P}\frac{\partial T_i}{\partial \eta_i} = \sin\Phi\,[(\tfrac{1}{2}\sin^2\Phi - \Theta) + \tfrac{1}{2}(\cos\Phi - W)^2],
\end{aligned}\right\} \tag{18}$$

$$\left.\begin{aligned}
&\rho_i = -\sin\Phi/V_i, \quad p_i = \rho_i\Theta,\\
&u_i = w_i(\xi_i) - \sin\Phi\,(\cos\Phi - W)\,\{\mathscr{Y}_i(\xi_i) - \eta_i\}/\Theta^\omega,\\
&T_i = S_i(\xi_i) - P\sin\Phi\,[(\tfrac{1}{2}\sin^2\Phi - \Theta) + \tfrac{1}{2}(\cos\Phi - W)^2]\,\{\mathscr{Y}_i - \eta_i\}/\Theta^\omega,\\
&\left|\sin^2\Phi\left(V_i + \frac{2\Theta}{\sin\Phi}\right)\right| = \exp\left[\frac{\sin\Phi}{2\Theta}\left(V_i + \frac{2\Theta}{\sin\Phi}\right) - \frac{3}{8}\frac{\sin^3\Phi}{\Theta^{1+\omega}}\{\mathscr{Y}_i - \eta_i\}\right].
\end{aligned}\right\} \tag{19}$$

In the course of finding these solutions it is found that

$$\left.\begin{aligned} w_0 &= w_m = -(\cos\Phi - \mathrm{W})/(\sin\Phi)^{\frac{4}{3}}, \\ S_0 &= S_m = \left[\Theta + \frac{P}{3-2P}\sin^2\Phi + \frac{P}{2(2-P)}(\cos\Phi - W)^2\right] \Big/ (\sin\Phi)^{\frac{4}{3}}, \end{aligned}\right\} \tag{20}$$

where W and Θ are determined from the shock-layer solution.

2.6. *The shock layer*

For the shock layer

$$x = \xi, \quad y = \epsilon\eta_L, \tag{21}$$

$$\left.\begin{aligned} &u = u_L + \ldots, \quad v = \epsilon v_L + \ldots, \\ &\rho = (1/\epsilon)\rho_L + \ldots, \quad p = (1/\epsilon\delta)p_L + \ldots, \quad T = (1/\delta)T_L + \ldots. \end{aligned}\right\} \tag{22}$$

From matching with the inner region of the shock structure, it is found that the boundary conditions at the outer edge of either an inviscid ($K \to 0$) or viscous ($K = O(1)$) shock layer, $\mathscr{Y}_L$, whose position is determined only after solution of the shock-layer flow, are

$$\left.\begin{aligned} &u_L(\xi, \mathscr{Y}_L) = W(\xi); \quad W_{\mathrm{inv}}(\xi) = \cos\Phi(\xi), \quad W_{\mathrm{visc}} \text{ to be determined}, \\ &T_L(\xi, \mathscr{Y}_L) = \Theta(\xi); \Theta_{\mathrm{inv}}(\xi) = \tfrac{1}{2}\sin^2\Phi(\xi), \ \Theta_{\mathrm{visc}} \text{ to be determined}, \\ &v_L(\xi, \mathscr{Y}_L) = -(2\Theta/\sin\Phi(\xi)) + \mathscr{Y}_L'(\xi)\, W(\xi), \\ &K\Theta^{\omega}(\partial u_L/\partial\eta_L)(\xi, \mathscr{Y}_L) = \sin\Phi(\cos\Phi - W), \\ &K\Theta^{\omega}(\partial T_L/\delta\eta_L)(\xi, \mathscr{Y}_L) = P\sin\Phi\, \tfrac{1}{2}[(\sin^2\Phi - 2\Theta) + (\cos\Phi - W)^2]. \end{aligned}\right\} \tag{23}$$

It should be noted that, once these outer edge boundary conditions are determined, the flow in the entire shock layer is found without further reference to the shock structure. That is, equations (23) are the appropriate 'shock conditions'. For an inviscid layer, K goes to zero and the usual Rankine–Hugoniot conditions are recovered. The shock conditions are essentially those given by Cheng (1961).

The reduced equations of motion for the inviscid or viscous shock layer are

$$\left.\begin{aligned} &p_L = \rho_L T_L, \\ &\frac{\partial}{\partial\xi}(B\rho_L u_L) + \frac{\partial}{\partial\eta_L}(B\rho_L v_L) = 0, \\ &2\frac{\partial p_L}{\partial\eta_L} - \kappa\rho_L u_L^2 = 0, \\ &\rho_L\left(u_L\frac{\partial u_L}{\partial\xi} + v_L\frac{\partial u_L}{\partial\eta_L}\right) = K\frac{\partial}{\partial\eta_L}\left(T_L^{\omega}\frac{\partial u_L}{\partial\eta_L}\right), \\ &\rho_L\left(u_L\frac{\partial T_L}{\partial\xi} + v_L\frac{\delta T_L}{\partial\eta_L}\right) = K\left[\frac{1}{P}\frac{\partial}{\partial\eta_L}\left(T_L^{\omega}\frac{\partial T_L}{\partial\eta_L}\right) + T_L^{\omega}\left(\frac{\partial u_L}{\partial\eta_L}\right)^2\right]. \end{aligned}\right\} \tag{24}$$

If $s = \xi$, $t_L = u_L/\cos\Phi = u_L/\sigma$, and $\tau_L = \{(T_L^{\omega}/B)(\partial u_L/\partial\eta_L)\}$, then the solutions for the flow in the inviscid shock layer, consistent with shock boundary conditions, are

$$\left.\begin{aligned} T_L &= \tfrac{1}{2}[1-t_L^2\sigma^2], \\ p_L &= \frac{1}{2}\left[(1-\sigma^2)-\frac{\kappa(\sigma)}{B(\sigma)}\int_{t_L\sigma}^{\sigma}\frac{\nu B(\nu)\,d\nu}{\kappa(\nu)}\right], \\ \tau_L &= \frac{t_L\sigma\kappa(t_L\sigma)}{B(t_L\sigma)}\frac{p_L}{T_L^{1-\omega}}, \\ \eta_L &= \frac{1}{B(\sigma)(1-\sigma^2)}\int_0^{t_L}\frac{(1-\sigma^2\nu^2)\,B(\sigma\nu)\,d\nu}{\nu\kappa(\sigma\nu)\left[1-\dfrac{\kappa(\sigma)}{B(\sigma)(1-\sigma^2)}\displaystyle\int_{\sigma\nu}^{\sigma}\frac{hB(h)\,dh}{\kappa(h)}\right]}. \end{aligned}\right\} \quad (25)$$

These solutions are equivalent to those found by Freeman (1956).

Complete solutions, however, are not presented for the viscous shock-layer equations since extensive numerical work would be necessary. However, the shear and heat transfer at the nose of the body for the viscous shock layer computed for the case of $\omega = \frac{1}{2}$, $P = \frac{3}{4}$, and a wall temperature that is zero are presented in figures 3 and 4.

2.7. *The body layer*

The appropriate variables in the body are

$$x = \xi, \quad y = \epsilon^{\frac{3}{2}}\eta_c, \quad (26)$$

$$\left.\begin{aligned} &u = \epsilon^{\frac{1}{2}}u_c+\ldots, \quad v = \epsilon^2 v_c+\ldots, \\ &\rho = (1/\epsilon)\rho_c+\ldots, \quad p = (1/\epsilon\delta)p_c+\ldots, \quad T = (1/\delta)T_c+\ldots. \end{aligned}\right\} \quad (27)$$

The leading terms of the inviscid or viscous equations of motion for this region are

$$\left.\begin{aligned} &p_c = \rho_c T_c, \\ &\frac{\partial}{\partial\xi}(B\rho_c u_c)+\frac{\partial}{\partial\eta_c}(B\rho_c v_c) = 0, \\ &\partial p_c/\partial\eta_c = 0, \\ &\rho_c\left(u_c\frac{\partial u_c}{\partial\xi}+v_c\frac{\partial u_c}{\partial\eta_c}\right)+2\frac{\partial p_c}{\partial\xi} = D\frac{\partial}{\partial\eta_c}\left(T_c^{\omega}\frac{\partial u_c}{\partial\eta_c}\right), \\ &\rho_c\left(u_c\frac{\partial T_c}{\partial\xi}+v_c\frac{\partial T_c}{\partial\eta_c}\right) = \frac{D}{P}\frac{\partial}{\partial\eta_c}\left(T_c^{\omega}\frac{\partial T_c}{\partial\eta_c}\right). \end{aligned}\right\} \quad (28)$$

From matching with the inviscid shock layer, the boundary conditions at the outer edge of the body layer, which are the same whether this layer is inviscid or viscous, are

$$\left.\begin{aligned} &p_c(\xi,\eta_c\to\infty) = \frac{1}{2}\left[\sin^2\Phi(\xi)-\frac{\kappa(\xi)}{B(\xi)}\int_0^{\cos\Phi(\xi)}\frac{\nu B(\nu)\,d\nu}{\kappa(\nu)}\right] \equiv p_c(\xi), \\ &T_c(\xi,\eta_c\to\infty) = \tfrac{1}{2}, \\ &(\partial u_c/\partial\eta_c)(\xi,\eta_c\to\infty) = 2B(\xi)\,p_c(\xi). \end{aligned}\right\} \quad (29a)$$

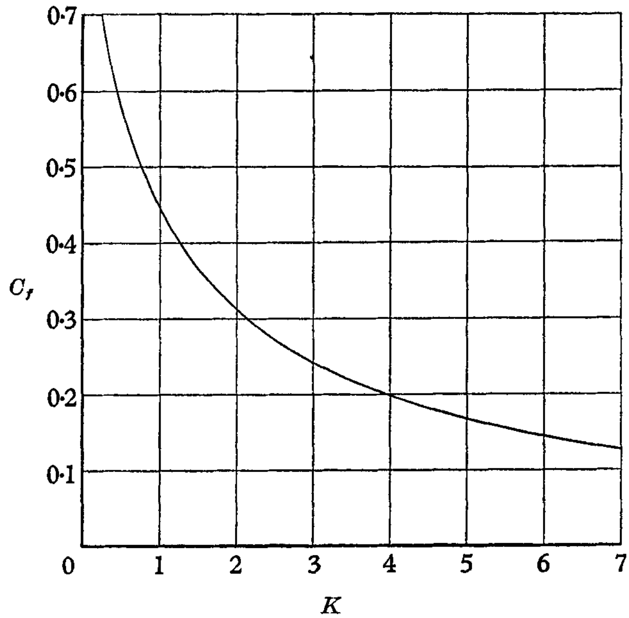

FIGURE 3. The skin friction coefficient $(C_f)_L = \lim_{s, t_L \to 0} (\tau_L)$ *vs* the similarity parameter K of the viscous shock layer.

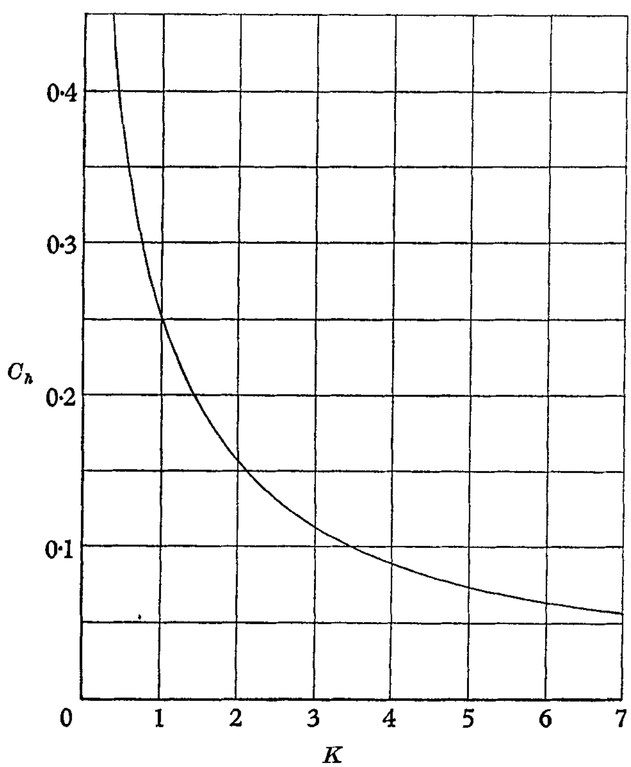

FIGURE 4. The heat transfer coefficient $(C_h)_L = \lim_{s, t_L \to 0} \left(\frac{\cos\Phi}{W} \tau_L \frac{\partial T_L}{\partial t_L}\right)$ *vs* the similarity parameter K of the viscous shock layer.

In Crocco variables ($s = \xi, t_c = u_c/\cos\Phi, \tau_c = \{(T_c^\omega/B)(\partial u_c/\partial\eta_c)\}$), these boundary conditions become

$$p_c(s, t_c \to \infty) = p_c(s), \quad T_c(s, t_c \to \infty) = \tfrac{1}{2}, \quad \tau_c(s, t_c \to \infty) = 2^{1-\omega} p_c(s). \quad (29b)$$

Using this second set of boundary conditions, the solutions for the inviscid body layer are found to be

$$\left.\begin{aligned} &p_c(s, t_c) = p_c(s), \quad T_c(s, t_c) = \tfrac{1}{2}, \quad \tau_c(s, t_c) = 2^{1-\omega} p_c(s), \\ &\eta_c(s, t_c) = \frac{\cos\Phi(s)}{2B(s)\, p_c(s)} [t_c - t_{c,w}(s)]; \quad t_{c,w}(s) = \frac{[-2\ln\{2p_c(s)\}]^{\frac{1}{2}}}{\cos\Phi(s)}. \end{aligned}\right\} \quad (30)$$

No solutions are presented for the viscous body layer; again numerical work would be necessary.

2.8. *The viscous boundary layer*

The quantities in the viscous boundary layer are

$$x = \xi, \quad y = (\epsilon^{\frac{1}{2}}/R\delta^\omega)^{\frac{1}{2}} \eta_{BL}, \quad (31)$$

$$\left.\begin{aligned} &u = \epsilon^{\frac{1}{2}} u_{BL} + \dots, \qquad v = (\epsilon^{\frac{3}{2}}/R\delta^\omega)^{\frac{1}{2}} v_{BL} + \dots, \\ &\rho = (1/\epsilon)\rho_{BL} + \dots, \quad p = (1/\epsilon\delta) p_{BL} + \dots, \quad T = (1/\delta) T_{BL} + \dots. \end{aligned}\right\} \quad (32)$$

The equations of motion for the region are

$$\left.\begin{aligned} &p_{BL} = \rho_{BL} T_{BL}, \\ &\frac{\partial}{\partial\xi}(B\rho_{BL} u_{BL}) + \frac{\partial}{\partial\eta_{BL}}(B\rho_{BL} v_{BL}) = 0, \\ &\partial p_{BL}/\partial\eta_{BL} = 0, \\ &\rho_{BL}\left(u_{BL}\frac{\partial u_{BL}}{\partial\xi} + v_{BL}\frac{\partial u_{BL}}{\partial\eta_{BL}}\right) + 2\frac{\partial p_{BL}}{\partial\xi} = \frac{\partial}{\partial\eta_{BL}}\left(T_{BL}^\omega \frac{\partial u_{BL}}{\partial\eta_{BL}}\right), \\ &\rho_{BL}\left(u_{BL}\frac{\partial T_{BL}}{\partial\xi} + v_{BL}\frac{\partial T_{BL}}{\partial\eta_{BL}}\right) = \frac{1}{P}\frac{\partial}{\partial\eta_{BL}}\left(T_{BL}^\omega \frac{\partial T_{BL}}{\partial\eta_{BL}}\right). \end{aligned}\right\} \quad (33)$$

Matching with the inviscid body layer yields the boundary conditions at the outer edge of the boundary layer, which are

$$\left.\begin{aligned} &T_{BL}(\xi, \eta_{BL} \to \infty) = \tfrac{1}{2}, \quad p_{BL}(\xi, \eta_{BL} \to \infty) = p_{BL}(\xi) = p_c(\xi), \\ &u_{BL}(\xi, \eta_{BL} \to \infty) = [-2\ln\{2p_c(\xi)\}]^{\frac{1}{2}}. \end{aligned}\right\} \quad (34)$$

No solutions are presented here for the viscous boundary layer. Problems of this type have been studied in great detail.

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